Lecture 2
Dehn surgery

2.1 Knots and links in 3-manifolds

A finite collection of smoothly embedded disjoint closed curves in a closed orientable 3-manifold M is called a ***link***. A one-component link is called a ***knot***. We will not distinguish between equivalent knots and links: two links, $\mathcal{L}$ and $\mathcal{L}'$, in M are said to be equivalent if there is a smooth orientation preserving automorphism $h\colon M \to M$ such that $h(\mathcal{L}) = \mathcal{L}'$. In case the links have two or more components, we also assign a fixed ordering of the components and require that h respect the orderings. Every link $\mathcal{L} \subset M$ can be thickened to get its ***tubular neighborhood*** $N(\mathcal{L})$ which is a collection of smoothly embedded disjoint solid tori, $D^2 \times S^1$, one for each link component, whose cores $\{0\} \times S^1$ form the link $\mathcal{L}$.

Links in $S^3 = \mathbb{R}^3 \cup \{\infty\}$ can be thought of as links in $\mathbb{R}^3$. The requirement that each of the curves of a link be smoothly embedded avoids pathological examples like the one pictured in Figure 2.1.

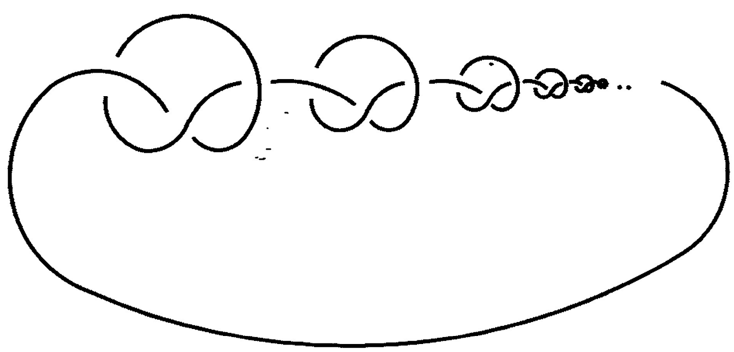


Figure 2.1

Let $\mathcal{L}$ be a link in $\mathbb{R}^3$ represented by a collection of disjoint smoothly embedded curves. Let P be a plane and $p\colon \mathbb{R}^3 \to P$ the orthogonal projection. We say that P is ***regular*** for the link $\mathcal{L}$ provided that every $p^{-1}(x)$, $x \in P$, intersects $\mathcal{L}$ in 0, 1 or 2 points and the Jacobian $d_y p$ has rank 1 at every intersection point $y \in p^{-1}(x)$. Every link admits a regular projection, see Crowell–Fox [33]. Thus links in $S^3 = \mathbb{R}^3 \cup \{\infty\}$ are often described by their regular projections, and drawn as smooth curves in $\mathbb{R}^2$ with marked undercrossings and overcrossings at each double point.

Any knot in S^3 equivalent to the knot $(\cos t, \sin t, 0)$, $0 \le t \le 2\pi$, is called a ***trivial knot*** or an ***unknot***.

2.2 Surgery on links in S^3

Let k be a knot in a closed orientable 3-manifold M, and $N(k)$ its tubular neighborhood. By cutting the manifold M open along the 2-torus $\partial N(k)$ we get two manifolds – one is the ***knot exterior*** K which is the closure of $M \setminus N(k)$, and the other is the solid torus $N(k)$ which we will identify with the standard solid torus $D^2 \times S^1$. Thus K is a manifold with boundary $\partial K = T^2$ and $M = K \cup (D^2 \times S^1)$. One can use an arbitrary homeomorphism $h\colon \partial D^2 \times S^1 \to \partial K$ to glue $D^2 \times S^1$ back in K. The space we obtain by this construction, $Q = K \cup_h (D^2 \times S^1)$, is a closed orientable 3-manifold. We say that Q is obtained from M by ***surgery*** along k.

The manifold Q depends on the choice of homeomorphism h. In fact, the manifold Q is completely determined by the image under h of the meridian $\partial D^2 \times \{*\}$ of the solid torus $D^2 \times S^1$, i.e. by the curve $c = h(\partial D^2 \times \{*\})$ on the boundary of K. To see this, one simply repeats the argument that used the Figure 1.10 from Lecture 1.

If $M = S^3$ then a curve on ∂K is given, up to isotopy, by a pair of relatively prime integers (p, q). The construction is as follows. The space K has integral homology groups $H_0(K) = H_1(K) = \mathbb{Z}$ and $H_i(K) = 0$ if $i \geq 2$. Any meridian of $N(k)$ represents a generator of $H_1(K)$; this is a curve on ∂K which we call m. Up to isotopy of $N(k)$, there is a unique longitude which is homologically trivial in K; this gives another curve, ℓ, on ∂K. These two form a basis for $H_1(\partial K)$ which is unique up to isotopy and reversing the orientations of m and ℓ. The longitude ℓ is called a ***canonical longitude*** to distinguish it from the longitude defined in Lecture 1.

We fix the orientations as follows. Choose the standard orientation on $S^3 = \mathbb{R}^3 \cup \{\infty\}$; it induces an orientation on K. We choose directions on the curves m and ℓ so that the triple $\langle m, \ell, n\rangle$ is positively oriented. Here, n is a normal vector to ∂K pointing inside K, see Figure 2.2.

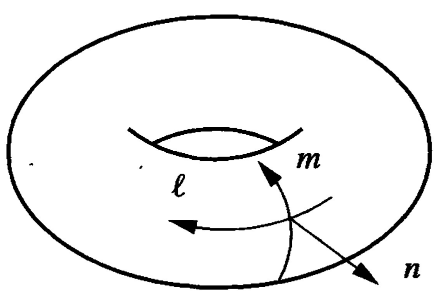


Figure 2.2

Any simple closed curve c on ∂K is now isotopic to a curve of the form $c = p\cdot m + q\cdot \ell$. The pairs (p, q) and $(-p, -q)$ define the same curve c since the orientation of c is of no importance to us. One can conveniently think of a pair (p, q) as a reduced fraction p/q. Then there is a one-to-one correspondence between the set of isotopy classes of non-trivial simple closed curves on the torus ∂K and the set of reduced fractions p/q. This set should be completed by $1/0 = \infty$, which corresponds to the meridian m. The result of $1/0$-surgery on any knot $k \subset S^3$ is again S^3.

Surgeries of the type described above are called ***rational***. A surgery is called ***integral*** if $q = \pm 1$. Similarly, one defines rational and integral surgeries along a link $\mathcal{L} \subset M$: the surgery along each link component should be rational, respectively, integral. In general, surgery along a knot $k \subset M$ cannot be described by a rational number since there is no canonical choice of the longitude (such a choice exists, however, for a homology 3-sphere M, see Section 7.5). Nevertheless, the concept of integral surgery still makes sense: the curve $\partial D^2 \times \{*\}$ on $D^2 \times S^1$ should be attached to a curve on ∂K running exactly once along *a* longitude.

Theorem 2.1 (Lickorish [95] and Wallace [145]). *Every closed orientable 3-manifold M can be obtained from S^3 by an integral surgery on a link $\mathcal{L} \subset S^3$.*

Lemma 2.2. *Let $h_1, h_2 \colon \partial H \to \partial H'$ be homeomorphisms of the surfaces of two handlebodies such that $h_1 = h_2\tau_c$ where τ_c is a twist along a simple closed curve $c \subset \partial H_1$. Then the manifold $M_2 = H \cup_{h_2} H'$ is obtained from the manifold $M_1 = H \cup_{h_1} H'$ by an integral surgery along a knot $k \subset M_1$ isotopic to the image of c.*

Proof of Lemma 2.2. We push the curve c inside the handlebody H to get a knot $k \subset H$. Let $N(k)$ be its tubular neighborhood, and $A \cong S^1 \times I$ an annulus connecting c and $\partial N(k)$, see Figure 2.3.

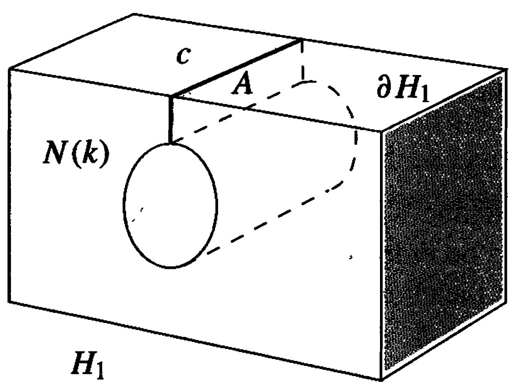


Figure 2.3

Let $\varphi \colon H \setminus N(k) \to H \setminus N(k)$ be a homeomorphism which cuts the space $H \setminus N(k)$ open along the annulus A, twists one of the rims by 360°, and glues it back in. The restriction of the homeomorphism φ to ∂H is the twist τ_c while its restriction to $\partial N(k)$ is a twist along the longitude $\ell = A \cap N(k)$ of the knot k. Let $M_i' = (H \setminus N(k)) \cup_{h_i} H'$, $i = 1, 2$. The formula

$$\Phi(x) = \begin{cases} \varphi(x), & \text{if } x \in H \setminus N(k), \\ x, & \text{if } x \in H', \end{cases} \tag{2.1}$$

defines a homeomorphism of M_2' to M_1'. The conditions $h_1 = h_2\tau_\ell$ and $\varphi|_{\partial H_1}$ assure that the two parts of the formula (2.1) agree on the boundary, see Figure 2.4.

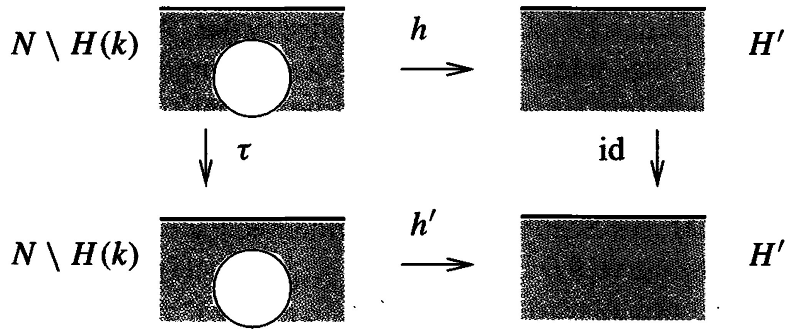


Figure 2.4

Thus, if we remove the solid tori corresponding to $N(k)$ from the manifolds M_1 and M_2, they become homeomorphic. This implies that M_2 is obtained from M_1 by surgery along the knot k. Since Φ maps the meridian m of the torus $\partial N(k)$ to the curve $m \pm \ell$, this surgery is integral. □

Proof of Theorem 2.1. Every manifold M can be represented as $M = H \cup_{h_2} H'$, where H and H' are handlebodies of genus g, and h_2 is an orientation reversing homeomorphism of their boundaries. Similarly, $S^3 = H \cup_{h_1} H'$. Therefore, $h_2^{-1} h_1$ is an orientation preserving homeomorphism so $h_2^{-1} h_1 = \tau_{c_1} \tau_{c_2} \ldots \tau_{c_n}$ where τ_{c_i} is a twist along a curve c_i. According to Lemma 2.2, multiplying the gluing homeomorphism by a Dehn twist has the same effect as performing an integral surgery along a knot. A sequence of such multiplications gives a sequence of surgeries on knots, or a surgery on a link. □

Thus, any closed orientable 3-manifold can be obtained by an integral surgery along a link $\mathcal{L} \subset S^3$. It should be emphasized again that the result of the surgery depends not only on $\mathcal{L}$ but also on the choice of simple closed curves in the boundary $\partial N(k)$ of each component k of the link $\mathcal{L}$. As we have seen, for an integral surgery, such a curve is uniquely determined by an integer. A choice of an integer for each component of $\mathcal{L}$ is called a ***framing*** of $\mathcal{L}$. A link $\mathcal{L}$ with a fixed framing will be called a ***framed link***.

2.3 Surgery description of lens spaces and Seifert manifolds

Let $p \geq 2$. The lens space $L(p, 1)$ can be obtained by gluing together two solid tori by the homeomorphism

$$\begin{pmatrix} -1 & 0 \\ p & 1 \end{pmatrix}$$

which attaches the meridian μ_1 of the first torus to the curve $-\mu_2 + p \cdot \lambda_2$ on the second, see Figure 2.5 where $p = 3$.

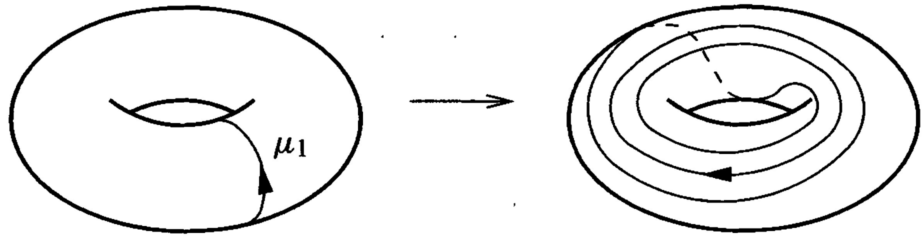


Figure 2.5

If we turn the second solid torus inside out and think of it as a trivial knot exterior, the meridian μ_1 will be attached to the curve $\ell - p \cdot m$. Thus $L(p, 1)$ has surgery description shown in Figure 2.6.

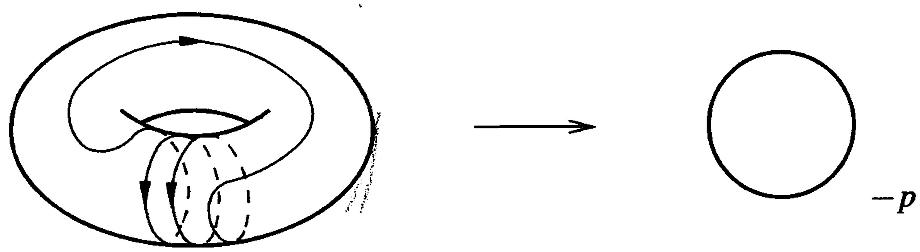


Figure 2.6

Similarly, any $L(p, q)$ will be a rational surgery on an unknotted circle with framing p/q. To produce $L(p, q)$ by an integral surgery, replace one of the solid tori $S^1 \times D^2$ by $S^1 \times \Delta^2$ where Δ^2 is an annulus. The construction above which produced $L(p, 1)$, will then give a manifold with boundary a torus. The latter can be pictured as a surgered solid torus as shown in Figure 2.7 ($L(p, 1)$ can be obtained from it by gluing in a solid torus by the identity map).

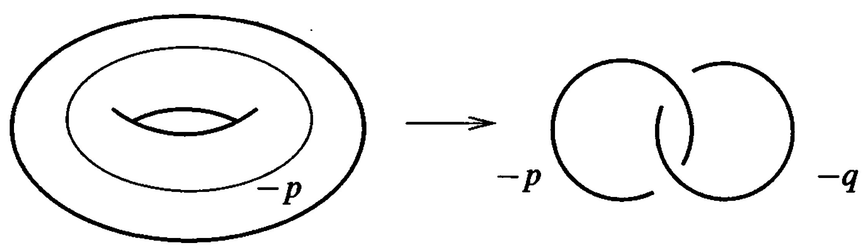


Figure 2.7

Repeat the construction with p replaced by any integer q relatively prime to p. Glue these two surgered solid tori together along their boundary by the homeomorphism

We obtain S^3 surgered along the link pictured in Figure 2.7. On the other hand,

$$\begin{pmatrix} -1 & 0 \\ p & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ q & 1 \end{pmatrix} = \begin{pmatrix} -q & -1 \\ pq-1 & p \end{pmatrix},$$

therefore, the link in Figure 2.7 represents $L(pq-1, q)$.

Theorem 2.3. *Any lens space $L(p,q)$ has a surgery description as in Figure 2.8, where $p/q = [x_1, \dots, x_n]$ is a continued fraction decomposition,*

$$[x_1, \dots, x_n] = x_1 - \cfrac{1}{x_2 - \cfrac{1}{\cdots - \cfrac{1}{x_n}}} \tag{2.2}$$

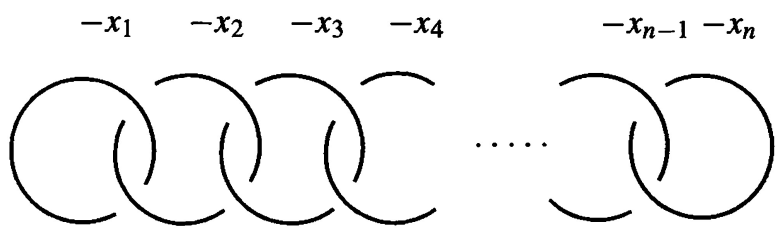


Figure 2.8

Proof. The construction for $L(pq-1, q)$ can be repeated sufficiently many times to produce the link in Figure 2.8. The only thing we need to check is that, if $p/q = [x_1, \dots, x_n]$, then

$$\begin{pmatrix} -q & s \\ p & r \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ x_1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ x_2 & 1 \end{pmatrix}\cdots\begin{pmatrix} -1 & 0 \\ x_n & 1 \end{pmatrix}$$

for some r and s. This is true for $n = 1$ and $n = 2$ because

$$\frac{p}{1} = [p] \quad \text{and} \quad \frac{pq-1}{q} = [p, q].$$

By induction, suppose that $p'/q' = [x_2, \dots, x_n]$, then

$$\begin{pmatrix} -1 & 0 \\ x_1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -q' & s' \\ p' & r' \end{pmatrix} = \begin{pmatrix} -p' & -r' \\ x_1 p' - q' & x_1 r' + s' \end{pmatrix},$$

so that

$$\frac{x_1 p' - q'}{p'} = x_1 - \frac{q'}{p'} = x_1 - \frac{1}{[x_2, \dots, x_n]} = [x_1, \dots, x_n].$$

Since every rational number has a continued fraction of the form described, we are finished. ☐

Figure 2.9

The link in Figure 2.8 is usually drawn as the weighted graph shown in Figure 2.9 where each vertex corresponds to an unknot, and two vertices are connected by an edge if the corresponding unknots are linked.

Example. The lens space $L(7, 3)$ is a surgery on each of the following links in Figure 2.10 according to the continued fraction decompositions $7/3 = [3, 2, 2]$ and $7/3 = [2, -3]$.

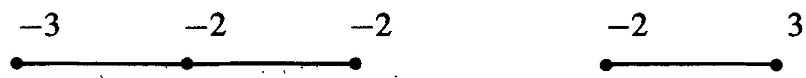


Figure 2.10

A Seifert manifold $M((a_1, b_1), \ldots, (a_n, b_n))$ has a rational surgery description shown in Figure 2.11. This description fixes an orientation of the manifold M. From now on, we will refer to $M((a_1, b_1), \ldots, (a_n, b_n))$ as an oriented 3-manifold with this particular orientation.

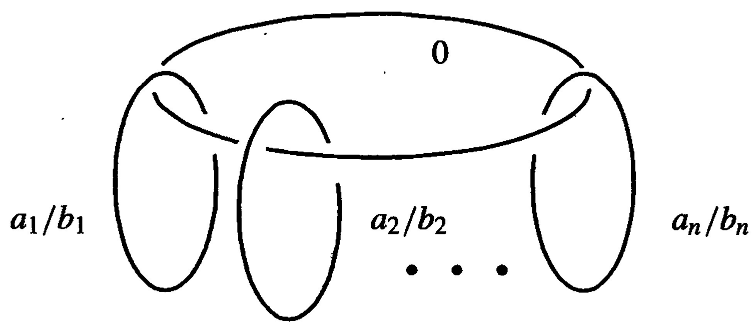


Figure 2.11

With the graph notations as above, the manifold $M((a_1, b_1), \ldots, (a_n, b_n))$ can be described as shown in Figure 2.12 where $a_i/b_i = [x_{i1}, \ldots, x_{im_i}]$.

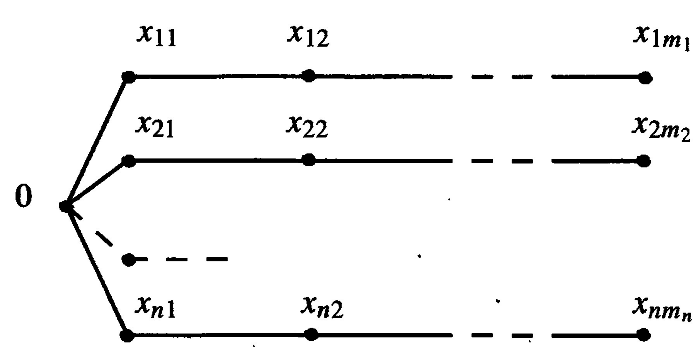


Figure 2.12

Example. The manifold $M((3, 2), (4, -1), (5, -2))$ has the surgery description shown in Figure 2.13.

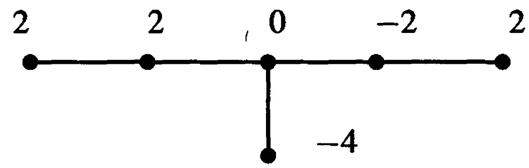


Figure 2.13

2.4 Surgery and 4-manifolds

An oriented compact smooth 4-dimensional manifold W is called an (oriented) ***cobordism*** between two closed oriented 3-manifolds M_1 and M_2 if $\partial W = -M_1 \cup M_2$ where $-M_1$ stands for M_1 with reversed orientation. If M_1 is empty, one says that M_2 is ***cobordant to zero***.

There is a closed relationship between surgeries on framed links and cobordisms. Let k be a knot in M with an (integral) framing defined by a curve c in ∂K such that $[c] = [k] \in H_1(N(k))$. Let a be a point on the boundary of D^2. Then there exists a unique (up to isotopy) diffeomorphism $h: S^1 \times D^2 \to N(k)$ such that $h(S^1 \times \{0\}) = k$ and $h(S^1 \times \{a\}) = c$. Glue a 2-handle $D^2 \times D^2$ to the 4-manifold $M \times [0, 1]$ with the help of the embedding $h: S^1 \times D^2 = (\partial D^2) \times D^2 \to N(k) \subset M = M \times \{1\}$. What we get is a 4-manifold $W = (M \times [0, 1]) \cup_h (D^2 \times D^2)$. It is called the ***trace*** of surgery on k.

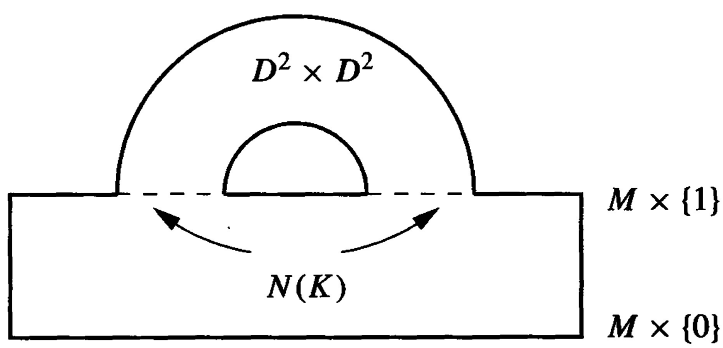


Figure 2.14

Theorem 2.4. *The manifold W is a cobordism between M and the manifold obtained from M by surgery on k.*

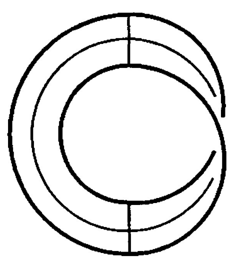
Proof. The boundary of W consists of two components. One of these, namely $M \times \{0\}$, is homeomorphic to M. Gluing $D^2 \times D^2$ to $M \times [0, 1]$ changes $M \times \{1\}$ as follows: the solid torus $N(K) = h(\partial D^2 \times D^2)$ is removed and replaced by the solid torus $D^2 \times \partial D^2$ (which is a "free" portion of the boundary $\partial(D^2 \times D^2)$). Note that the meridian $\partial D^2 \times \{a\}$ is identified with the curve $c = h(\partial D^2 \times \{a\})$. This means that an integral surgery is performed on $M \times \{1\}$ along k with the framing given by c. Formally speaking, the manifold W is not smooth as it has "corners" after gluing in the handle. However, there is a canonical way to provide W with the structure of a smooth manifold. One "smoothes out" the corners using techniques described, e.g., in Chapter 1 of Conner–Floyd [32]. □

Corollary 2.5. *Any closed oriented* 3*-manifold is cobordant to zero.*

Proof. Any closed oriented 3-manifold M can be obtained by an integral surgery on a link in S^3. Theorem 2.4 then implies that M is cobordant to S^3 which, in its turn, bounds a 4-ball. Therefore, M is cobordant to zero. □

Example. For any p, the lens space $L(p, 1)$ is a surgery on the link shown in Figure 2.6. The corresponding 4-manifold $E_p = D^4 \cup (D^2 \times D^2)$ with boundary $\partial E_p = L(p, 1)$ can be thought of as a union of $D^4 \cong D^2 \times D^2$ and a 2-handle $D^2 \times D^2$ glued along $S^1 \times D^2 \subset \partial(D^2 \times D^2)$ by a certain homeomorphism $h\colon S^1 \times D^2 \to S^1 \times D^2$. The homeomorphism h attaches $S^1 \times \{0\}$ to $S^1 \times \{0\}$ and twists a copy of D^2 p times in the counter-clockwise direction as one completes one circle along S^1. Schematically, this can be pictured as in Figure 2.15.

$D^4 = D^2 \times D^2$



$D^2 \times D^2$

Figure 2.15

The central discs of both handles $D^2 \times D^2$ are glued along S^1 to produce a copy of S^2 inside E_p.

Example. For every p, the manifold E_p is a locally trivial bundle over S^2 with the fiber D^2. The manifold E_0 is a trivial bundle, i.e. a product $E_0 = S^2 \times D^2$. Its boundary is $\partial E_0 = \partial(S^2 \times D^2) = S^2 \times S^1$.

Example. The manifold E_1 can be identified with a punctured complex projective plane $\mathbb{C}P^2 \setminus D^4$, so that $\partial E_1 = \partial(\mathbb{C}P^2 \setminus D^4) = S^3$. Before we prove this, we recall that, by definition,

$$\mathbb{C}P^2 = \{ (z_0, z_1, z_2) \in \mathbb{C}^3 \setminus 0 \}/\mathbb{C}^*$$

where $\mathbb{C}^*$ is the multiplicative group of non-zero complex numbers acting by the rule $(z_0, z_1, z_2) \mapsto (cz_0, cz_1, cz_2)$, $c \in \mathbb{C}^*$. The equivalence class of (z_0, z_1, z_2) is usually denoted by $[z_0 : z_1 : z_2]$.

The complex projective plane $\mathbb{C}P^2$ is covered by three coordinate charts $U_i = \{ z_i \neq 0 \}$, $i = 0, 1, 2$, each of which is homeomorphic to $\mathbb{C}^2$ via homeomorphisms

$$\begin{aligned} h_0\colon U_0 \to \mathbb{C}^2, \qquad & [z_0 : z_1 : z_2] \mapsto (z_1/z_0, z_2/z_0), \\ h_1\colon U_1 \to \mathbb{C}^2, \qquad & [z_0 : z_1 : z_2] \mapsto (z_0/z_1, z_2/z_1), \\ h_2\colon U_2 \to \mathbb{C}^2, \qquad & [z_0 : z_1 : z_2] \mapsto (z_0/z_2, z_1/z_2). \end{aligned}$$

The charts U_0 and U_1 together cover all of $\mathbb{C}P^2$ but the point $[0 : 0 : 1]$. Therefore, $U_0 \cup U_1$ is a punctured $\mathbb{C}P^2$. Both $h_0(U_0 \cap U_1)$ and $h_1(U_0 \cap U_1)$ as subsets of $\mathbb{C}^2$ consist of all points (z, w) with $z \neq 0$ so that $h_0(U_0 \cap U_1) = h_1(U_0 \cap U_1) = S^1 \times \mathbb{R}_+ \times \mathbb{C}$. The gluing map

$$h_1(U_0 \cap U_1) \xrightarrow{h_1^{-1}} U_0 \cap U_1 \xrightarrow{h_0} h_0(U_0 \cap U_1)$$

is given by the formula $(z, w) \mapsto (z^{-1}, wz^{-1})$. The points (z, w) with $|z| \geq 1$ are mapped by this map to the points (z, w) with $|z| \leq 1$. Thus one can truncate $h_0(U_0)$ by the condition $|z| \leq 1$ and $h_1(U_1)$ by the condition $|z| \geq 1$, and think of the gluing operation as happening along $S^1 \times D^2$, where S^1 is given by $|z| = 1$, according to the map $(z, w) \mapsto (z^{-1}, wz^{-1})$. This is the map describing the manifold E_1.

As a complex manifold, $\mathbb{C}P^2$ comes with a canonical orientation. More careful analysis shows that in fact E_1 is diffeomorphic via an orientation preserving diffeomorphism to $\overline{\mathbb{C}P}^2 \setminus D^4$ where $\overline{\mathbb{C}P}^2$ stands for the complex projective plane with reversed orientation, and that $E_{-1} = \mathbb{C}P^2 \setminus D^4$. In short, the results of the last two examples can be formulated as in Figure 2.16.

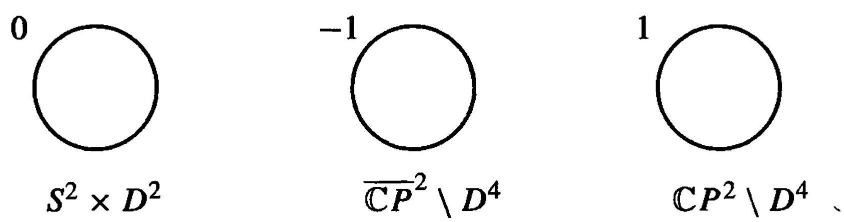


Figure 2.16